A New Proof of a Theorem in Analysis by Generating Integrals and Fractional Calculus

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Abstract

The idea of generating integrals analogous to generating functions is first introduced in this paper. A new proof of the well-known Finite Harmonic Series Theorem in Analysis and Analytical Number Theory is then obtained by the method of Generating Integrals and Fractional Calculus. A generalization of the Riemann zeta function up to non-integer order is derived.

1 The Finite Harmonic Series Theorem

Definition 1 *The finite harmonic series is*

$$h(n) = \sum_{k=1}^{n} \frac{1}{k} = 1 + \frac{1}{2} + \cdots + \frac{1}{n}\ .$$

Theorem 1 (The Finite Harmonic Series Theorem)
It is well known [1, p. 16, (1.7.9)] that

$$\begin{aligned} h(n) &= \psi(1{+}n) + \gamma && (1)\\ &= \log n + \gamma + O(1/n) && (2) \end{aligned}$$

where the digamma function $\psi(z) = \dfrac{d}{dz}\log\left(\Gamma(z)\right) = \dfrac{\Gamma'(z)}{\Gamma(z)}$, *the Euler constant* $\gamma = -\,\Gamma'(1) = 0.577215\cdots$, *and the prime denotes differentiation.*

The first part (1) of the Theorem can be proved by differentiating the recurrence relation of $\Gamma(n{+}1) = n\,\Gamma(n)$ and summing over n [2, p. 256]. The second part (2) can be proved by using the Euler-Maclaurin summation formula [4, p. 629].

In this paper, we shall present an alternative elementary proof [3] of the first part (1) of the Theorem by using **Fractional Calculus** and the idea of **generating integrals**.

2 Introduction & Motivation of Fractional Calculus

> *"Thus it follows that $d^{1/2}x$ will be equal to ... from which one day useful consequences will be drawn."* — Leibniz in a letter [5] to L'Hospital.

> *"When n is an integer, the ratio d^np, p a function of x, to dx^n can always be expressed algebraically. Now it is asked: what kind of ratio can be made if n is a fraction?"* — Euler [6].

> *"The idea of an integral or derivative, of arbitrary non-integral order, was introduced into analysis by Liouville and Riemann. Such integrals and derivatives may be, and have been by different writers, defined in a variety of manners, and different systems of definitions may be the most useful in different fields of analysis."* — Hardy and Littlewood [7].

In this paper, we shall consider D^σ, $\sigma \in \mathsf{R}$, an operator of non-integer order, a notion first pondered upon by Leibniz [5, pp. 301-302], Euler [6, p. 55], Lagrange [8], Laplace [9, p. 85 and p. 186], Fourier [10] and Abel [11] during the late 17th century to early 19th century. We shall briefly review several known and equally valid definitions for D^σ, and then focus on one of the definitions, the Riemann-Liouville (R-L) Fractional Calculus. The foundation of R-L Fractional Calculus was laid by Riemann [12] and Liouville [13] in the late 19th century, and then subsequently developed by Cayley [14], Laurent [15], Heaviside [16], Hardy and Littlewood [7, 17, 18, 19], and many others. It was largely regarded as a mathematical curiosity until only recently when Mandelbrot, the discoverer of Fractals, found an application of the R-L Fractional Calculus in the Brownian motion in a fractal medium, and speculated a possible connection between the analysis of Fractional Calculus and the geometry of Fractals [20].

We shall also introduce the idea of generating integrals by analogy to generating functions. As we shall see, just as a certain generating function is useful for generating a certain desired sequence of numbers, a certain generating integral is similarly useful. While a generating function $f(z)$ generates a sequence of numbers $\{p_n\}$ in the coefficients of the terms of different orders in its power series expansion $\sum_n p_n z^n$, a generating integral generates a sequence of numbers q_n in the coefficient of a term in the result of an n-fold integration.

However, a generating integral has one unique advantage over a generating function. The R-L Fractional Calculus can be used to analytically extend a generating integral of iteration order $n \in \mathsf{Z}^+$ to order $\rho \in \mathsf{R}$. The result is that the sequence of numbers $\{q_n\}$ is in turn analytically extended to a function $q(\rho)$, $\rho \in \mathsf{R}$.

We shall then show how the Riemann-Liouville Fractional Calculus and the idea of generating integrals can be used to prove the well-known Finite Harmonic Series Theorem.

3 Differential-Integral Operator D^n

Definition 2 *Let the operator $D^n_{x|a}$, $n \in \mathsf{Z}$, acting on a function f at the point x be defined as*

$$D^n_{x|a} f(x) = \begin{cases} \dfrac{d^n}{dx^n} f(x) & (n > 0) \\[2ex] f(x) & (n = 0) \\[2ex] \displaystyle\int_a^x f(\hat{x})(d\hat{x})^{-n} & (n < 0) \end{cases} \tag{3}$$

where the n-fold integration is defined inductively as

$$\int_a^x f(\hat{x})(d\hat{x})^{-n} = \underbrace{\int_a^x \int_a^{x_n} \int_a^{x_{n-1}} \cdots \int_a^{x_3} \int_a^{x_2}}_{n\text{-times}} f(x_1)\, dx_1\, dx_2 \cdots dx_{n-1}\, dx_n \ .$$

As an example, consider $f(x) = x^m$.

$$D^n_{x|a} x^m = \begin{cases} 0 & (m \geq 0,\ m < n) \\[2ex] \dfrac{m!}{(m-n)!}\, x^{m-n} = \dfrac{\Gamma(1+m)}{\Gamma(1+m-n)}\, x^{m-n} & (m \geq 0,\ m \geq n) \\[2ex] (-1)^n \dfrac{(|m|-1+n)!}{(|m|-1)!}\, \hat{x}^{m-n}\Big|_a^x & (m \leq 0,\ m < n) \\ \quad = \displaystyle\lim_{\epsilon \to 0} \frac{\Gamma(1+m+\epsilon)}{\Gamma(1+m+\epsilon-n)}\, \hat{x}^{m-n}\Big|_a^x & \\[2ex] \displaystyle\int_a^x \left(\int_a^{\hat{x}} \tilde{x}^m (d\tilde{x})^{|m|} \right) (d\hat{x})^{(m-n)} & (m \leq 0,\ m \geq n) \\ \quad = (-1)^{m+1} \dfrac{1}{(|m|-1)!} \displaystyle\int_a^x \log \hat{x} (d\hat{x})^{(m-n)} & \end{cases} \tag{4}$$

where $\hat{x}^{m-n}\Big|_a^x = (x^{m-n} - a^{m-n})$.

If we tabulate $D^n_{x|a} x^m$ for $n, m \in \mathsf{Z}$ and omit the constant terms containing a, we can observe a pattern emerges as in Table 1.

$D_x^n x^m$							
$m \backslash n$	-3	-2	-1	0	1	2	3
2	2!/5! x^5	2!/4! x^4	2!/3! x^3	x^2	2! x	2!	0
1	1/4! x^4	1/3! x^3	1/2! x^2	x	1	0	0
0	1/3! x^3	1/2! x^2	x	1	0	0	0
-1	$\int \log x\,(dx)^2$	$\int \log x\,(dx)$	$\log x$	x^{-1}	- x^{-2}	2! x^{-3}	-3! x^{-4}
-2	-$\int \log x\,(dx)$	- $\log x$	- x^{-1}	x^{-2}	-2! x^{-3}	3! x^{-4}	-4! x^{-5}
-3	1/2! $\log x$	1/2! x^{-1}	-1/2! x^{-2}	x^{-3}	-3!/2! x^{-4}	4!/2! x^{-5}	-5!/2! x^{-6}

Table 1: Tabulated results of $D_x^n x^m$

4 Riemann-Liouville (R-L) Fractional Calculus

The R-L Fractional Calculus [21] begins with

$$\int_a^x f(\hat{x})\,(d\hat{x})^n \;\equiv\; \underbrace{\int_a^x \int_a^{x_n} \int_a^{x_{n-1}} \cdots \int_a^{x_3} \int_a^{x_2}}_{n\text{-times}} f(x_1)\,dx_1\,dx_2 \cdots dx_{n-1}\,dx_n \tag{5}$$

for $n \in \mathsf{Z}+$ as the fundamental defining expression, and it can be shown [21, p. 38] to be equal to the Cauchy formula for repeated integration,

$$\frac{1}{\Gamma(n)} \int_a^x \frac{f(t)}{(x-t)^{1-n}}\,dt\;. \tag{6}$$

Definition 3 (R-L Fractional Calculus)
The R-L fractional integral is analytically extended from (6) as

$$\begin{aligned} D_{x|a}^{\sigma} f(x) &= \frac{d^\sigma}{dx^\sigma} f(x) = \int_a^x f(x)(dx)^{-\sigma} \quad \textit{by extending } (3) \\ &= \frac{1}{\Gamma(-\sigma)} \int_a^x \frac{f(t)}{(x-t)^{1+\sigma}}\,dt \quad (\sigma < 0,\; \sigma, a \in \mathsf{R}) \;\; \textit{by } (6)\;, \end{aligned} \tag{7}$$

and the R-L fractional derivative is in turn derived from the R-L fractional integral (7) by ordinary differentiation:

$$D_{x|a}^{\sigma} f(x) \;=\; D_{x|a}^{m} \left(D_{x|a}^{-(m-\sigma)} f(x) \right) \quad (\sigma > 0,\; m \in \mathsf{Z}^+) \tag{8}$$

where m is chosen such that $m > 1+\sigma,\ \sigma > 0$.

Lemma 1 *The equation (8) is independent of the choice of m for $m > 1+\sigma,\ m \in \mathsf{Z}^+,\ \sigma \in \mathsf{R},\ \sigma > 0$.*

Proof

For $m > 1{+}\sigma,\ m \in \mathsf{Z}^+,\ \sigma > 0$, we have $-(m{-}\sigma) < -1 < 0$ and $(m{-}\sigma{-}1) > 0$. The first condition, $-(m-\sigma) < 0$, allows us to use the equation (7) to write

$$D_{x|a}^{-(m-\sigma)} f(x) \ = \ \frac{1}{\Gamma(m-\sigma)} \int_a^x \frac{f(t)}{(x{-}t)^{1+\sigma-m}}\, dt \ .$$

From (8),

$$\begin{aligned} & D_{x|a}^{m} \left(D_{x|a}^{-(m-\sigma)} f(x) \right) \\ = \ & \frac{d^m}{dx^m} \left(\frac{1}{\Gamma(m{-}\sigma)} \int_a^x \frac{f(t)}{(x{-}t)^{1+\sigma-m}}\, dt \right) \\ = \ & \frac{1}{\Gamma(m{-}\sigma)} \int_a^x f(t) \left(\frac{d^m}{dx^m} (x{-}t)^{m-\sigma-1} \right) dt \ . \end{aligned} \tag{9}$$

The second condition, $(m{-}\sigma{-}1) > 0$, and the condition $m > 0$ allow us to use the second case of (8). Thus, (9) becomes

$$\begin{aligned} & \frac{1}{\Gamma(m{-}\sigma)} \int_a^x f(t)\, \frac{\Gamma(m{-}\sigma)}{\Gamma(-\sigma)}\, (x{-}t)^{-(1+\sigma)}\, dt \\ = \ & \frac{1}{\Gamma(-\sigma)} \int_a^x \frac{f(t)}{(x{-}t)^{1+\sigma}}\, dt \\ = \ & D_{x|a}^{\sigma} f(x) \ . \end{aligned}$$

□

When $a = 0$, (7) for $f(x) = x^r$ is well-defined only for the half plane $r > -1$. Consequently, in the R-L Fractional Calculus, $D_{x|a}^{\sigma} x^r$ is well-defined only for the half plane $r > -1$.

$$D_{x|a}^{\sigma} x^r = \begin{cases} \dfrac{\Gamma(1{+}r)}{\Gamma(1{+}r{-}\sigma)}\, x^{r-\sigma} & (\sigma > 0,\ r > -1) \\ x^r & (\sigma = 0,\ \forall\, r) \\ \dfrac{\Gamma(1{+}r)}{\Gamma(1{+}r{-}\sigma)}\, \hat{x}^{r-\sigma}\Big|_a^x & (\sigma < 0,\ r > -1) \end{cases} \quad . \tag{10}$$

5 Fractional Calculus by Cauchy Integral

The Cauchy Integral for an analytic function $f(z)$ in the complex plane [22, p. 120] is

$$f^{(n)}(z_0) \ = \ \frac{\Gamma(1+n)}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^{1+n}}\, dz \ . \tag{11}$$

Analytic extension of the Cauchy Integral from $n \in \mathsf{Z}^+$ to $s \in \mathsf{R}^+$ gives an analytic extension of $D^n f(z_0)$. However, the analytic extension is not trivial. The term $(z-z_0)^{1+\sigma}$ will become multi-valued and thus the result may depend on the choice of branch cut and integration path.

6 Fractional Calculus by Fourier Transform

In the theory of Fourier Transforms,

$$\begin{aligned}\tilde{f}(x) &= \int_{-\infty}^{+\infty} f(x)\, e^{ikx}\, dx \,, \\ f(x) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \tilde{f}(x)\, e^{-ikx}\, dk \,,\end{aligned}$$

and

$$\begin{aligned}D_x^\sigma f(x) &= \int_{-\infty}^{+\infty} \tilde{f}(k)\, D_x^\sigma \left(e^{-ikx}\right) dk \quad (\sigma \in \mathsf{R}) \\ &= \int_{-\infty}^{+\infty} (-ik)^\sigma \tilde{f}(k)\; e^{-ikx}\, dk \,.\end{aligned}$$

This approach is often known as the pseudo-differential operator approach. It was shown by Závada [23] to be equivalent to the Riemann-Liouville Fractional Calculus and the Fractional Calculus by Cauchy Integral.

7 Functional Analytic Approach

In the functional analytic approach, an example of a functional integral of an operator A is

$$(-A)^a = -\frac{\sin a\pi}{\pi}\int_0^\infty \lambda^{a-1}\,(\lambda\mathbb{1} - A)^{-1} A\; d\lambda \quad (0 < a < 1,\ \lambda \in \mathsf{R}) \,. \tag{12}$$

$(\lambda\mathbb{1} - A)$ is called the kernel of the functional integral. The evaluation of the integral with respect to real variable λ requires various conditions on the spectrum of the operator A.

The analytic extension of D in the functional approach is then obtained from replacing A by D in (12).

For details of this well-developed functional analysis approach, see [24].

8 Differentiating and Integrating in non-integer s-dimensions

The differential of an integer n-dimensional function in n-dimensions can be expressed as

$$\frac{\partial}{\partial x_1}\frac{\partial}{\partial x_2}\cdots\frac{\partial}{\partial x_n} f(x_1, x_2, \cdots, x_n) \,.$$

The corresponding integral can be expressed as

$$\int f(x_1, x_2, \cdots, x_n)\, d^n x \equiv \underbrace{\int^{x_n}\!\!\cdots\int^{x_2}\!\!\int^{x_1}}_{n\text{-times}} f(x_1, x_2, \cdots, x_n)\, dx_1 dx_2 \cdots dx_n \,.$$

If f is spherically symmetric, $f = f(r)$, then

$$\frac{\partial}{\partial x_1}\frac{\partial}{\partial x_2}\cdots\frac{\partial}{\partial x_n} f = \frac{\partial^{n-1}}{\partial r^{n-1}}\frac{\partial}{\partial \Omega_{n-1}} f \equiv \frac{\Gamma(n/2)}{2\pi^{n/2}}\frac{\partial^{n-1}}{\partial r^{n-1}} f(r) \tag{13}$$

can then be analytically extended to the differential of a non-integer s-dimensional function in s-dimensions,

$$\frac{\partial^{s-1}}{\partial r^{s-1}}\frac{\partial}{\partial \Omega_{s-1}} f(r) = \frac{\Gamma(s/2)}{2\pi^{s/2}}\frac{\partial^{s-1}}{\partial r^{s-1}} f(r) \tag{14}$$

where $s \in \mathsf{R}$ or $s \in \mathsf{C}$, and Ω is the n-dimensional solid angle.

Similarly, the corresponding integral

$$\begin{aligned}\int f\, d^n x &\equiv \int_0^\infty r^{n-1} f(r)\, dr \int_0^{2\pi} d\theta_1 \int_0^\pi \sin\theta_2 d\theta_2 \cdots \int_0^\pi \sin^{n-2}\theta_{n-1} d\theta_{n-1} \\ &= \int_0^\infty r^{n-1} f(r)\, dr \int d\Omega_{n-1} = \frac{2\pi^{n/2}}{\Gamma(n/2)} \int_0^\infty r^{n-1} f(r)\, dr \end{aligned} \tag{15}$$

can be analytically extended to

$$\int f\, d^s x = \frac{2\pi^{s/2}}{\Gamma(s/2)} \int_0^\infty r^{s-1} f(r)\, dr\ . \tag{16}$$

This method was developed by 't Hooft and Veltman [25] in 1960's. The method was central to an important technique called Dimensional Regularization in Quantum Field Theory where the method is used to isolate singularities in divergent integrals.

9 Generating Integral of the Finite Harmonic Series

Theorem 2 (Generating Integral of $h(n)$)

$$\boxed{h(n) = \log x - \frac{\Gamma(1+n)}{x^n}\int_0^x \log\hat{x}\,(d\hat{x})^n \quad (n \in \mathsf{Z})} \tag{17}$$

where

$$\int_a^x f(\hat{x})\,(d\hat{x})^n \equiv \underbrace{\int_a^x\int_a^{x_n}\int_a^{x_{n-1}}\cdots\int_a^{x_3}\int_a^{x_2}}_{n\text{-times}} f(x_1)\, dx_1\, dx_2 \cdots dx_{n-1}\, dx_n\ .$$

Proof

We observe that $-h(n)/n!$ appears in the coefficient of the x^n term when we repeatedly integrate $\log x$:

$$\begin{aligned}\int_0^x \log\hat{x}\,(d\hat{x}) &= x(\log x - 1)\ , \\ \int_0^x \log\hat{x}\,(d\hat{x})^2 &= \frac{x^2}{2}(\log x - \frac{3}{2})\ , \\ &\ \vdots \\ \int_0^x \log\hat{x}\,(d\hat{x})^n &= \frac{x^n}{n!}(\log x - h(n))\ . \end{aligned} \tag{18}$$

We can prove this observation by induction:

$$\begin{aligned}
\int_0^x \log\hat{x}\,(d\hat{x})^{n+1} &= \int_0^x \left(\int_0^{\hat{x}} \log\tilde{x}\,(d\tilde{x})^n\right)(d\hat{x}) \\
&= \int_0^x \frac{\hat{x}^n}{n!}(\,\log\hat{x} - h(n)\,)(d\hat{x}) \\
&= \frac{x^{n+1}}{(n+1)!}(\,\log x - h(n)\,) - \int_0^x \frac{\hat{x}^n}{(n+1)!}(d\hat{x}) \\
&= \frac{x^{n+1}}{(n+1)!}\left(\log x - h(n) - \frac{1}{n+1}\right) \\
&= \frac{x^{n+1}}{(n+1)!}(\,\log x - h(n+1)\,)\ .
\end{aligned}$$

Rearrangement of (18) yields the Theorem. □

Theorem 3 (Generating Integral of $h(\rho)$)

$$\boxed{h(\rho) = \log x - \frac{\Gamma(1+\rho)}{x^\rho}\int_0^x \log\hat{x}\,(d\hat{x})^\rho \quad (\rho \in \mathsf{R})}\ . \tag{19}$$

Proof

By analogy to generating functions, we take

$$\int_0^x \log\hat{x}\,(d\hat{x})^n$$

as the *generating integral* of the finite harmonic series $h(n)$, and so the natural analytic extension of the generating integral takes the form of

$$\int_0^x \log\hat{x}\,(d\hat{x})^\rho = \frac{x^\rho}{\Gamma(1+\rho)}\,(\log x - h(\rho))\ . \tag{20}$$

Noting that $\log x$ may be expressed as

$$\log x = \int_1^x \hat{x}^{-1}d\hat{x} = \lim_{\epsilon\to 0}\int_1^x \hat{x}^{-1+\epsilon}\,d\hat{x} = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\,(x^\epsilon - 1)\ , \tag{21}$$

we can now evaluate the fractional integral in (20) by the **R-L Fractional Calculus** (Definition 3).

$$\begin{aligned}
\int_0^x \log\hat{x}\,(d\hat{x})^\rho &= \int_0^x \lim_{\epsilon\to 0}\frac{1}{\epsilon}\,(\hat{x}^\epsilon - 1)\,(d\hat{x})^\rho = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_0^x (\hat{x}^\epsilon - 1)\,(d\hat{x})^\rho \\
&= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left[\int_0^x \hat{x}^\epsilon(d\hat{x})^\rho - \int_0^x 1\,(d\hat{x})^\rho\right] \\
&= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left[D_{\hat{x}}^{-\rho}\hat{x}^\epsilon\Big|_0^x \ - \ D_{\hat{x}}^{-\rho}\,1\,\Big|_0^x\right] \\
&= \lim_{\epsilon\to 0}\frac{x^\rho}{\epsilon}\left[\frac{\Gamma(1+\epsilon)\,x^\epsilon}{\Gamma(1+\epsilon+\rho)} - \frac{1}{\Gamma(1+\rho)}\right]
\end{aligned} \tag{22}$$

where the interchange of the integral and the limit is justified by Arzelà's theorem on bounded convergence [26, pp. 405-406] as $(\hat{x}^{\epsilon}-1)/\epsilon$ is integrable in $\hat{x}\in[0,x]$.

Combining (22) with (20) gives the analytic extension of the finite harmonic series:

$$\begin{aligned}
h(\rho) &= \log x - \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left[\frac{\Gamma(1+\epsilon)\,\Gamma(1+\rho)}{\Gamma(1+\epsilon+\rho)}\,x^{\epsilon}-1\right] \\
&= \log x - \lim_{\epsilon\to 0}\frac{1}{\epsilon}\,(x^{\epsilon}-1)\left[\frac{\Gamma(1+\epsilon)\,\Gamma(1+\rho)}{\Gamma(1+\epsilon+\rho)}\right] \\
&\qquad + \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left[1-\frac{\Gamma(1+\epsilon)\,\Gamma(1+\rho)}{\Gamma(1+\epsilon+\rho)}\right] \\
&= \left[\log x - \lim_{\epsilon\to 0}\frac{1}{\epsilon}\,(x^{\epsilon}-1)\right] + \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left[1-\frac{\Gamma(1+\epsilon)\,\Gamma(1+\rho)}{\Gamma(1+\epsilon+\rho)}\right] \\
&= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left[1-\frac{\Gamma(1+\epsilon)\,\Gamma(1+\rho)}{\Gamma(1+\epsilon+\rho)}\right] \\
&= \frac{\Gamma'(1+\rho)}{\Gamma(1+\rho)} - \Gamma'(1) \\
&= \psi(1+\rho)+\gamma \qquad (23)
\end{aligned}$$

where the limit has been taken with L'Hospital rule. □

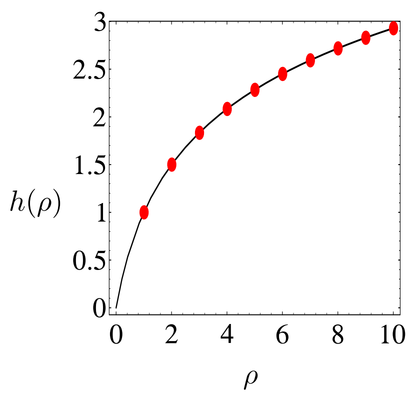

Figure 1: The curve $h(\rho)=\psi(1+\rho)+\gamma$ passes through the points $(n,h(n))$ where $n\in\mathsf{Z}^{+}$, $\mathsf{h}(\mathsf{n})=\sum_{\mathsf{k}=1}^{\mathsf{n}}\frac{1}{\mathsf{k}}$.

Hence, by the application of the R-L Fractional Calculus to analytically extend the generating integral, we have found an alternative elementary proof of the first part (1) of the Finite Harmonic Series Theorem.

Theorem 4

$$\boxed{\int_0^x \log\hat{x}\,(d\hat{x})^{\rho} = \frac{x^{\rho}}{\Gamma(1+\rho)}\,(\log x-\psi(1+\rho)-\gamma) \quad (\rho\in\mathsf{R})}\,. \qquad (24)$$

Proof

Replacing the $h(\rho)$ in (20) by (23) gives the Theorem. $\square$

10 The Riemann Zeta Function up to Order n

The analytic extension (23) can be generalized.

Definition 4 *The Riemann zeta function [27]*

$$\zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s} \quad (\mathsf{Re}(s) > 1), \tag{25}$$

the polygamma functions [2, p. 260, (6.4.1)]

$$\psi^{(m)}(x) = \frac{d^m}{dx^m}\psi(x) = \frac{d^{m+1}}{dx^{m+1}} \log\Gamma(x) = (-1)^{m+1} m! \sum_{k=0}^{\infty} \frac{1}{(x+k)^{m+1}}\ , \tag{26}$$

and the Riemann zeta function up to order n

$$\zeta(s \mid n) = \sum_{k=1}^{n} \frac{1}{k^s} = \zeta(s) - \sum_{k=n+1}^{\infty} \frac{1}{k^s} \quad (\mathsf{Re}(s) > 1) \tag{27}$$

may be combined to write

$$\begin{aligned} \zeta(m \mid n) &= \sum_{k=1}^{n} \frac{1}{k^m} = \frac{(-1)^m}{(m-1)!} \left(\psi^{(m-1)}(1+n) - \psi^{(m-1)}(1)\right) \\ &= \frac{(-1)^m}{(m-1)!} \frac{d^m}{dx^m} \log(\Gamma(1+x)) \Big|_{x=0}^{x=n} . \end{aligned} \tag{28}$$

The analytic extension is then obtained by replacing the derivative in (28) with a fractional derivative:

$$\zeta(s \mid z) = \frac{w(s)}{\Gamma(s)}\, D_x^s \log(\Gamma(1+x)) \Big|_{x=0}^{x=z} \quad (s, z \in \mathsf{C}) \tag{29}$$

which can be evaluated when $\log(\Gamma(1+x))$ is expressed in the form of an asymptotic series [2, p. 257, (6.1.41)]. However, $w(s)$ depends on the choice of extension to the **R-L Fractional Calculus** (Definition 3) into the other half plane, $r \leq -1,\ \sigma \in \mathsf{R}$.

11 Analytic Extension of R-L Fractional Calculus

Consider the case of D_x^n where $a = 0$. We shall introduce the (σ, r) diagram in which the numerical factor of $D_x^\sigma x^r$ is mapped to the point at coordinate (σ, r) of the diagram. The (σ, r) diagram of $D_x^\sigma x^r$ can be characterised into 4 regions as in Figure 2:

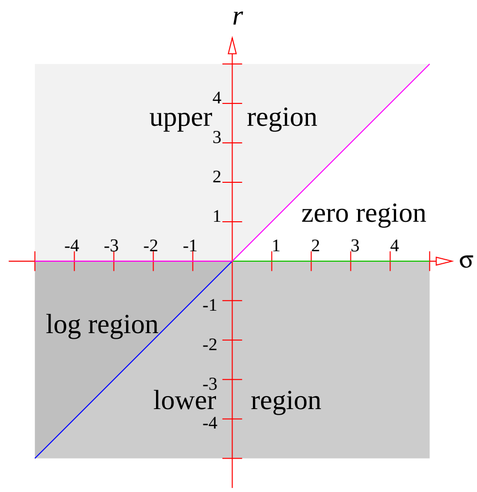

Figure 2: (σ, r) diagram of $D_x^\sigma x^r$.

Definition 5 (Regions of (σ, r))

The zero region	Z_{ero}	$= \{(\sigma, r) :\ r < \sigma,\ r \geq 0\}$;
the upper region	U_{pp}	$= \{(\sigma, r) :\ r \geq \sigma,\ r \geq 0\}$;
the lower region	L_{ow}	$= \{(\sigma, r) :\ r < \sigma,\ r < 0\}$;
the log region	L_{og}	$= \{(\sigma, r) :\ r \geq \sigma,\ r < 0\}$.

A point lying on the right of the r-axis ($\sigma > 0$) is a differentiation; a point on the left ($\sigma < 0$) is an integration.

The (σ, r) diagram at integer grid points everywhere except in the log region gives numerical factors identical to those in Table 1.

An extension of R-L Fractional Calculus to the other half plane $r \leq -1$ is given by

$$D_x^\sigma x^r = \begin{cases} \lim\limits_{\epsilon \to 0} \dfrac{\Gamma(1+r+\epsilon)}{\Gamma(\epsilon)} D_x^{\sigma - r} \log x & \text{in } \Omega \\ \lim\limits_{\epsilon \to 0} \dfrac{\Gamma(1+r+\epsilon)}{\Gamma(1+r+\epsilon-\sigma)} x^{r-\sigma} & \text{elsewhere} \end{cases} \tag{30}$$

where $\Omega = \{(\sigma, r) :\ \sigma \in \mathsf{R},\ \mathsf{r} \in \mathsf{Z}^-\}$, the set of horizontal lines in lower and log regions.

$\Gamma(1+r)/\Gamma(1+r-\sigma)$ is finite everywhere in the zero and upper regions. $\overset{\lim}{\epsilon \to 0}\ \Gamma(1+r+\epsilon)/\Gamma(1+r+\epsilon-\sigma)$ is well-defined everywhere in the lower and log regions except in $\Omega \backslash (\mathsf{Z}^- \times \mathsf{Z}^-)$. Following Theorem 4, the R-L fractional integral of $\log x$ can be evaluated exactly and expressed in only elementary functions,

$$D_x^\sigma \log x = \begin{cases} \dfrac{x^{-\sigma}}{\Gamma(1-\sigma)} \left(\log x - \psi(1-\sigma) - \gamma \right) & (\sigma \in \mathsf{R} \backslash \mathsf{Z}) \\ \lim\limits_{\epsilon \to 0} \dfrac{\Gamma(1+r+\epsilon)}{\Gamma(1+r+\epsilon-\sigma)} x^{r-\sigma} & (\sigma \in \mathsf{Z}) \end{cases} . \tag{31}$$

(30) is thus well-defined.

To analytically extend from $D_x^\sigma x^r$ on the real plane (σ, r) to $D_x^s x^r$ on the product of complex plane and real line, $(s, r) \in \mathsf{C} \times \mathsf{R}$, we simply replace $\sigma \in \mathsf{R}$ in (30) by $s \in \mathsf{C}$.

12 Open Problems

1. Generalize (30) for $D_{z-c}^s z^w$, $s, w, z, c \in \mathsf{C}$. For the complex function z^w, one has to specify, in addition, the integration contour for $\mathsf{Re}(s) < 0$.

2. Find the exact expression for $\log(\Gamma(1+x))$ and $w(s)$ in terms of elementary functions in analytic extension of the R-L Fractional Calculus given by (30).

13 Tables of Generating Integrals

"Nature laughs at the difficulties of integration." — Laplace

An interesting consequence is that objects of the form $\int x^r (\log x)^a (dx)^\rho$, $r, a, \rho \in \mathsf{R}$, exist and can be generating integrals for certain functions. Perhaps it may be worthwhile to introduce, in the future editions of Tables of Integrals, a new section which gives the coefficient of the x^k term, $w(\rho, r, a, k)$, corresponding to these generating integrals to facilitate the evaluation of integrals of similar forms.

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